BREATHING VIBRATIONS OF PRESSURIZED PARTIALLY FILLED TANKS

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Introduction

The problem of vibrations of a tank containing one or more fluids under pressure is of fundamental interest in a variety of problems of present day technology. A prime example of this is the design of large fuel tanks for liquid propellant rockets, where the requirement of a light weight structure and the large percentage of the total mass contributed by the propellant, call for as accurate an analysis as possible of the dynamic interaction between fluid and elastic container.

The breathing vibrations of a pressurized cylindrical shell containing a heavy liquid have been investigated by Berry and Reissner[1] using shallow shell theory. Lindholm et al[2] carried on an experimental investigation of partially filled cylindrical tanks. Chu[3] analyzed a cylindrical shell partially filled with an incompressible fluid using Donnell's[4] equations as extended by Yu[5] to the dynamic case and, following Reissner[6], he neglected the axial and circumferential inertial terms.

In the present paper the differential equations of motion are first established, within the scope of the linear small displacement theory, for an axisymetric pressurized shell filled with one or more non-mixable, non-viscous, compressible fluids. Thereafter, the specific case is considered of a closed simply supported circular cylindrical shell, a) completely filled with a single fluid and b) containing two non-mixable fluids.

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Equations of Motion

a) Shell

Within the scope of the linear elastic theory the differential equations of motion of a thin shell can be written[7, 8]

$$\sum_{j=1}^{3}\left[L_{ij}-\delta_{ij}m_s\frac{\partial^2}{\partial t^2}\right]u_j=p_i \quad (i=1,2,3) \qquad (1)$$

where:

u_j is the displacement component in the j-direction,

p_i is the component of the external force in the i-direction,

m_s is the inertial mass per unit area of the shell,

L_{ij} are linear differential operators in the shell coordinates associated with the shell geometry,

t is time,

δ_{ij} (Kronecker's delta) has the value 1 for $i = j$ and zero otherwise.

The natural modes are obtained as the solution of the set of 3 homogeneous differential equations

$$\sum_{j=1}^{3}\left[L_{ij}+\delta_{ij}m_s\omega_k^2\right]U_{jk}=0 \qquad (2)$$

obtained by substituting

$$u_{jk}=U_{jk}\sin\omega_k t \qquad (3)$$

into equation (1).

The functions U_{jk}, satisfying the homogeneous system (2) and associated boundary conditions, characterize the mode shape associated with the frequency ω_k.

The requirement that equations (2) have non-trivial solutions leads to the frequency equation associated with the problem.

b) Fluid

The small motion of a compressible non-viscous fluid are governed by the equation[9]

$$\left[\frac{1}{c_o^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right]q=0 \qquad (4)$$

where:

$q=p-p_o$ is the pressure fluctation about the mean value p_o,

c_o is the velocity of sound in the fluid at the pressure p_o,

∇^2 is the Laplacian operator,

The acceleration of a fluid particle in the x_i-direction is given by

$$\frac{\partial u_i}{\partial t}=-\frac{1}{\rho_o}\frac{\partial q}{\partial x_i} \qquad (5)$$

where ρ_o is the density of the fluid and u_i is the component of velocity in the x_i-direction.

c) Interaction Between Fluid and Shell

The interaction between the fluid and the shell is obtained by equating the normal accelerations and pressures of fluid and shell at the shell surface. Thus

$$\frac{\partial^2}{\partial t^2} u_n^s = \frac{\partial^2}{\partial t^2} u_n^f = -\frac{1}{\rho_o}\frac{\partial q}{\partial n} \tag{6}$$

$$p_n^s = p_n^f \tag{7}$$

at the shell surface. The indicies s and f refer to shell and fluid, and n denotes the normal direction.

If there are two non-mixable fluids inside the shell then, under the assumption that the dynamic overpressure as well as the pressure due to gravity are small as compared with the static pressure p_o, the following condition must be satisfied at the interface

$$u_n^{f_1} = u_n^{f_2} \tag{8}$$

which by equation (5) implies

$$\frac{1}{\rho_1}\frac{\partial q_1}{\partial n_1} = \frac{1}{\rho_2}\frac{\partial q_2}{\partial n_1} \tag{9}$$

where n_1 is the direction normal to the interface.

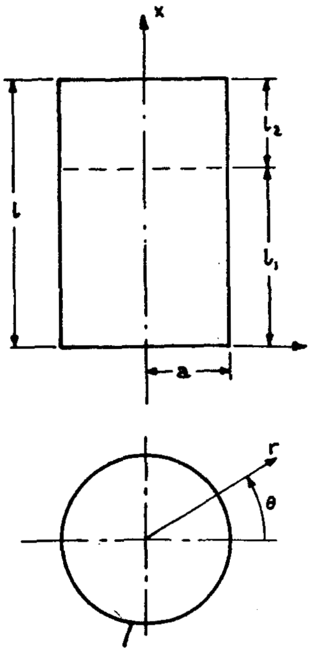

Fig. 1 Coordinate System

The Completely Filled Cylindrical Tank

For a cylindrical shell the differential equations (1) can be written as[7, 8]

$$\left[\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2a^2}\frac{\partial^2}{\partial\theta^2} - \frac{m_s}{D}\frac{\partial^2}{\partial t^2}\right]u + \frac{1+\nu}{2a}\frac{\partial^2}{\partial x\partial\theta}v - \frac{\nu}{a}\frac{\partial}{\partial x}w = 0$$

$$\frac{1+\nu}{2a}\frac{\partial^2}{\partial x\partial\theta}u + \left[\frac{1-\nu}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2}{\partial\theta^2} - \frac{m_s}{D}\frac{\partial^2}{\partial t^2}\right]v - \frac{1}{a^2}\frac{\partial}{\partial\theta}w = 0 \tag{10}$$

$$\frac{\nu}{a}\frac{\partial}{\partial x}u + \frac{1}{a^2}\frac{\partial}{\partial\theta}v - \left[\frac{1}{a^2} + \frac{h^2}{12}\nabla^4 - \frac{p_o a}{2D}\left(\frac{\partial^2}{\partial x^2} + \frac{2}{a^2}\frac{\partial^2}{\partial\theta^2}\right) + \frac{m_s}{D}\frac{\partial^2}{\partial t^2}\right]w = \frac{q}{D}$$

where:

u, v, w, are the displacement components in the x, θ, and r directions (see Figure 1),

a, h, m_s, are the radius, wall thickness, and mass per unit area of shell,

ν, $D = Eh/(1-\nu^2)$, are Poisson's ratio, and membrane stiffness respectively,

p_o is the static pressure, and q the dynamic overpressure acting on the shell.

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + \frac{2}{a^2}\frac{\partial^4}{\partial x^2\partial\theta^2} + \frac{1}{a^4}\frac{\partial^4}{\partial\theta^4}$$

For the simply supported shell the solutions of the associated homogeneous equations are of the form

$$\begin{aligned} u_{mn} &= U_{mn}\cos\frac{m\pi x}{l}\cos n\theta \sin\omega_{mn}t \\ v_{mn} &= V_{mn}\sin\frac{m\pi x}{l}\sin n\theta \sin\omega_{mn}t \\ w_{mn} &= W_{mn}\sin\frac{m\pi x}{l}\cos n\theta \sin\omega_{mn}t \end{aligned} \tag{11}$$

where l is the length of the cylindrical shell, and U_{mn}, V_{mn} and W_{mn} are constant coefficients. The fluid equation (4) in cylindrical coordinates has a solution of the form

$$q_{mn} = Q_{mn}J_n\left(\lambda\frac{r}{a}\right)\sin\frac{m\pi x}{l}\cos n\theta \sin\omega_{mn}t \tag{12}$$

where the J_n is the Bessel function of the first kind and order n, and

$$\lambda^2 = \lambda_{mn}^2 = \left(\frac{\omega_{mn}a}{c_o}\right)^2 - \left(\frac{m\pi a}{l}\right)^2 = \left(\frac{c_s}{c_o}\right)^2\Omega^2 - \beta^2 \tag{13}$$

with

$$\beta = \frac{m\pi a}{l} \; ; \; \Omega = \frac{\omega_{mn} a}{c_s} \; ; \; c_s = \sqrt{\frac{D}{m_s}} \tag{14}$$

Substituting (11) and (12) into (6) and (7) we obtain

$$q_{mn} = \omega_{mn}^2 a \rho_o B(\lambda) W_{mn} \sin\frac{m\pi x}{l} \cos n\theta \sin\omega_{mn} t \tag{15}$$

$$B(\lambda) = B_n = \frac{J_n(\lambda)}{\lambda J_n'(\lambda)} \tag{16}$$

where the prime indicates differentiation with respect to the argument. Substituting equation (15) and (11) into (10) we obtain

$$\begin{aligned}
(\Omega^2 - a_{11}) U_{mn} + a_{12} V_{mn} + a_{13} W_{mn} &= 0 \\
a_{21} U_{mn} + (\Omega^2 - a_{22}) V_{mn} + a_{23} W_{mn} &= 0 \\
a_{31} U_{mn} + a_{32} V_{mn} + \left([1+\mu B]\Omega^2 - a_{33}\right) W_{mn} &= 0
\end{aligned} \tag{17}$$

where:

$$a_{11} = \beta^2 + \frac{1-\nu}{2} n^2 ; \qquad a_{12} = a_{21} = \frac{1+\nu}{2}\beta n$$

$$a_{22} = \frac{1-\nu}{2}\beta^2 + n^2 ; \qquad a_{13} = a_{31} = -\nu\beta$$

$$a_{33} = 1 + \frac{\rho_o a}{2D}(2n^2+\beta^2) + \frac{h^2}{12a^2}(n^2+\beta^2)^2 ; \qquad a_{23} = a_{32} = n$$

$$\mu = \frac{a\rho_o}{m_s}$$

The requirement that (17) has a non-trivial solution leads to the characteristic equation

$$\begin{aligned}
&\left[(1+\mu B)\Omega^2 - a_{33}\right]\left[\Omega^2 - (n^2+\beta^2)\right]\left[\Omega^2 - \frac{1-\nu}{2}(n^2+\beta^2)\right] \\
&\quad - (n^2+\nu^2\beta^2)\Omega^2 + \frac{1-\nu}{2}(n^4 + 2n^2\beta^2 + \nu^2\beta^4) = 0
\end{aligned} \tag{18}$$

where:

$$a_{33} = 1 + \varkappa(2n^2+\beta^2) + \varepsilon(n^2+\beta^2)^2 ; \quad \varkappa = \frac{\rho_o a}{2D} \; ; \quad \varepsilon = \frac{h^2}{12a^2} \tag{19}$$

For the axisymetric case (n = 0) equation (18) reduces to

$$\left[(1+\mu B)\Omega^2 - a_{33}\right](\Omega^2 - \beta^2) - \nu^2\beta^2 = 0 \tag{18'}$$

with

$$a_{33} = 1 + \varkappa\beta^2 + \varepsilon\beta^4 \tag{19'}$$

The solution of the problem requires that equations (16) and (18) be simultaneously satisfied.

The Partially Filled Cylindrical Tank

Let the index i = 1, 2 indicate the bottom and top parts of the cylindrical shell as shown in Figure 1. The differential equations (1) can then be written as

$$\begin{aligned}
&\left[\frac{\partial^2}{\partial x^2} + \frac{1-\nu}{2a^2}\frac{\partial^2}{\partial\theta^2} - \frac{m_s}{D}\frac{\partial^2}{\partial t^2}\right]u_i + \frac{1+\nu}{2a}\frac{\partial^2}{\partial x\partial\theta}v_i - \frac{\nu}{a}\frac{\partial}{\partial x}w_i = 0 \\
&\frac{1+\nu}{2a}\frac{\partial^2}{\partial x\partial\theta}u_i + \left[\frac{1-\nu}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{a^2}\frac{\partial^2}{\partial\theta^2} - \frac{m_s}{D}\frac{\partial^2}{\partial t^2}\right]v_i - \frac{1}{a^2}\frac{\partial}{\partial\theta}w_i = 0 \\
&\frac{\nu}{a}\frac{\partial}{\partial x}u_i + \frac{1}{a^2}\frac{\partial}{\partial\theta}v_i - \left[\frac{1}{a^2} + \frac{h^2}{12}\nabla^4 - \frac{\rho_o a}{2D}\left(\frac{\partial^2}{\partial x^2} + \frac{2}{a^2}\frac{\partial^2}{\partial\theta^2}\right) + \frac{m_s}{D}\frac{\partial^2}{\partial t^2}\right]w_i = \frac{q_i}{D}
\end{aligned} \tag{20}$$

where the meaning of the symbols is the same as for the completely filled tank.

For the simply supported shell the solutions of the associated homogeneous equations can be assumed in the form of

$$\begin{aligned}
u_1 &= U_1 \cos\beta_1\frac{x}{a}\cos n\theta \sin\omega t \\
v_1 &= V_1 \sin\beta_1\frac{x}{a}\sin n\theta \sin\omega t \\
w_1 &= W_1 \sin\beta_1\frac{x}{a}\cos n\theta \sin\omega t
\end{aligned} \tag{21a}$$

$$\begin{aligned}
u_2 &= U_2 \cos\beta_2\frac{l-x}{a}\cos n\theta \sin\omega t \\
v_2 &= V_2 \sin\beta_2\frac{l-x}{a}\sin n\theta \sin\omega t \\
w_2 &= W_2 \sin\beta_2\frac{l-x}{a}\cos n\theta \sin\omega t
\end{aligned} \tag{21b}$$

subject to the conditions that at x = l_1

$$u_1 = u_2 \; ; \; v_1 = v_2 \; ; \; w_1 = w_2 \; ; \; \frac{\partial}{\partial x}w_1 = \frac{\partial}{\partial x}w_2 \tag{22}$$

Equation (22) yields

$$\beta_1 \cot\beta_1\frac{l_1}{a} + \beta_2 \cot\beta_2\frac{l_2}{a} = 0 \tag{23}$$

As before we assume solutions of the fluid equation (4) of the form

$$q_1 = Q_1 J_n(\lambda_1 \tfrac{r}{a}) \sin\beta_1 \tfrac{x}{a} \cos n\theta \sin\omega t \tag{24a}$$

$$q_2 = Q_2 J_n(\lambda_2 \tfrac{r}{a}) \sin\beta_2 \tfrac{L-x}{a} \cos n\theta \sin\omega t \tag{24b}$$

where:

$$\lambda_1^2 = \left(\frac{\omega a}{c_1}\right)^2 - \beta_1^2 = \left(\frac{c_s}{c_1}\right)^2 \Omega^2 - \beta_1^2 \tag{25a}$$

$$\lambda_2^2 = \left(\frac{\omega a}{c_2}\right)^2 - \beta_2^2 = \left(\frac{c_s}{c_2}\right)^2 \Omega^2 - \beta_2^2 \tag{25b}$$

$$\Omega^2 = \left(\frac{\omega a}{c_s}\right)^2 \tag{25c}$$

where c_1 and c_2 are the velocities of sound of the respective fluids. The satisfaction of equation (9) for all values of r requires that

$$\lambda_1 = \lambda_2 = \lambda \tag{26}$$

Substituting (21) and (24) into (6) and (7) we obtain

$$q_1 = \omega^2 a \rho_1 B(\lambda) W_1 \sin\beta_1 \tfrac{x}{a} \cos n\theta \sin\omega t \tag{27a}$$

$$q_2 = \omega^2 a \rho_2 B(\lambda) W_2 \sin\beta_2 \tfrac{L-x}{a} \cos n\theta \sin\omega t \tag{27b}$$

$$B(\lambda) = B_b = \frac{J_n(\lambda)}{\lambda J_n'(\lambda)} \tag{28}$$

Substitution of equation (27) into (20) leads finally to the pair of characteristic equations

$$\left[(1+\mu_i B)\Omega^2 - a_i\right]\left[\Omega^2 - (n^2+\beta_i^2)\right]\left[\Omega^2 - \tfrac{1-\nu}{2}(n^2+\beta_i^2)\right] - (n^2+\nu^2\beta_i^2)\Omega^2 + \tfrac{1-\nu}{2}(n^4+2n^2\beta_i^2+\nu^2\beta_i^4) = 0 \tag{29}$$

where:

$$a_i = 1 + \varkappa(2n^2+\beta_i^2) + \epsilon(n^2+\beta_i^2)^2 \tag{30}$$

and $\varkappa$, ϵ are defined in equation (19).

In the axisymetric case (n = 0) equations (29) and (30) reduced to

$$\left[(1+\mu_i B)\Omega^2 - a_i\right](\Omega^2 - \beta_i^2) - \nu^2\beta_i^2 = 0 \tag{29'}$$

$$a_i = 1 + \varkappa\beta_i^2 + \epsilon\beta_i^4 \tag{30'}$$

The solution of the problem requires that equations (23), (28) and (29) be simultaneously satisfied.

Numerical Examples

I) Full Tank

For the axisymetric case we write equation (18') in the form

$$\Omega_k^2 = \frac{1}{1+\mu B_{k-1}}\left[a_{33} + \frac{\nu^2\beta^2}{\Omega_{k-1}^2 - \beta^2}\right] \tag{31a}$$

or

$$\Omega_k^2 = \beta^2\left[1 + \frac{\nu^2}{(1+\mu B_{k-1})\Omega_{k-1}^2 - a_{33}}\right] \tag{31b}$$

with

$$B_k = B(\lambda_k) = -\frac{J_0(\lambda_k)}{\lambda_k J_1(\lambda_k)} \tag{32}$$

$$\lambda_k^2 = \left(\frac{c_s}{c_0}\right)^2 \Omega_k^2 - \beta^2 \tag{33}$$

let

$$\frac{L}{a} = 4; \quad \nu^2 = 0.1; \quad a_{33} = 1.001; \quad \mu = 150$$

$$m = 1; \quad \beta = \frac{m\pi a}{L} = 0.7854$$

a) To determine the first natural frequency we start with $\Omega_0 = 0$ and compute λ_0 and B_0 from equations (33) and (32). We then substitute these values into the right hand side of equation (31a) to compute Ω_1. We repeat these cycles until there is no further change. For the numerical values given, we obtain after three iteration cycles

$$\Omega = 0.1230$$

b) To determine the second natural frequency we start with $\Omega_0 = \beta$ and proceed as before using equation (31b) instead of (31a). After two iterations we obtain

$$\Omega = 0.61843$$

Figure 2 shows the graphical solution for this case.

II) Partially Filled Tank

Let

$$\varkappa = 1\times10^{-3};\quad \varepsilon = 5\times10^{-7};\quad \mu_1 = 150;\quad \mu_2 = 0.15;$$

$$\frac{c_1}{c_s} = 0.300;\quad \frac{c_3}{c_s} = 0.075;\quad \frac{L}{a} = 2;$$

Figure 3 shows the curves B_a corresponding to solutions of equations (29') and the curves B_b representing equation (28). The intersections of both sets of curves are the solutions of the system (29') and (28). In the same figure the ratios

$$\frac{\beta_1}{\Omega};\quad \frac{\beta_2}{\Omega};\quad \frac{\lambda}{\Omega};$$

are also plotted as functions of Ω. An iteration procedure similar to the one outlined above yielded for the first intersection

$$\Omega = 1.005;\quad \lambda = 3.196;\quad \beta_1 = 1.004;\quad \beta_2 = 13.012$$

Figure 4 shows the solution of the continuity equation (23) for the above given values of β_1, β_2 and l/a. The pairs of values l_1, l_2 so obtained determine the different liquid levels for which the system will vibrate at the frequency

$$\omega = \Omega\sqrt{\frac{D}{m_s a^2}} = 1.005\sqrt{\frac{D}{m_s a^2}}$$

Conclusions

The empty tank has three (two for $n = 0$) natural frequencies associated with a given mode shape, whereas, for the completely filled tank there can be any number of such natural frequencies. Moreover, the lowest natural frequency is always lower and the highest is at most equal to the corresponding natural frequencies of the empty tank.

When two fluids are present in the shell we have that, for the axisymmetric case, the frequencies are grouped into a low and a high range. The low range frequencies are lower than the lowest and the high range frequencies higher than the highest natural frequencies of the corresponding empty shell.

The case of n other than zero has not been investigated thoroughly but it is believed that similar conclusions can be drawn.

For a given circumferential mode shape there are in general several levels of the separation surface associated with the same natural frequency. However, if the liquid level is also prescribed there may not exist a corresponding natural frequency.

This property of the solution deserves further investigation, for it may very well imply, that when the liquid level changes, instability conditions which may induce sloshing are set up.

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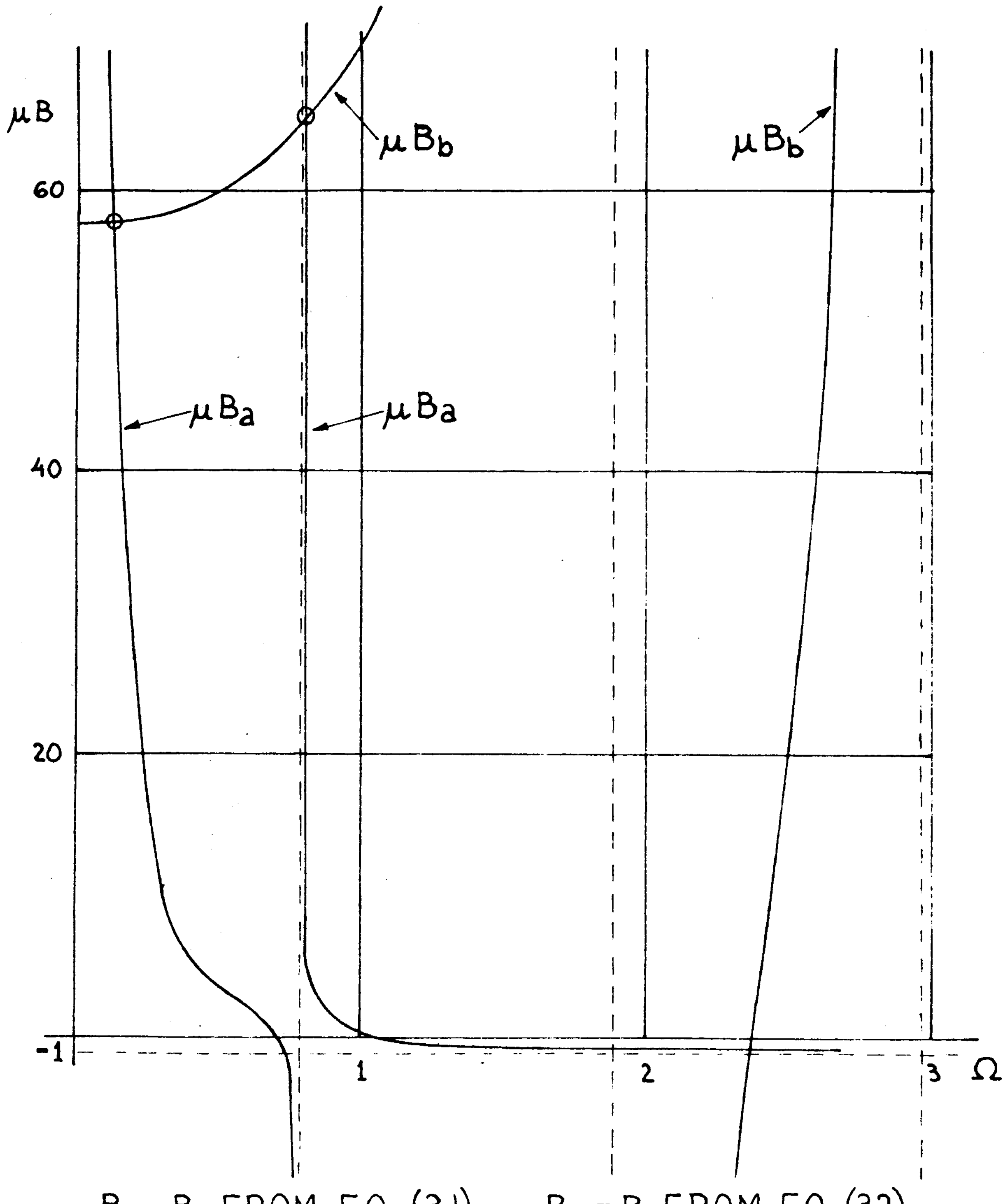

$B_a = B$ FROM EQ. (31) $B_b = B$ FROM EQ. (32)

FIG. 2 FULL TANK — SOLUTION FOR THE AXISYMMETRIC CASE

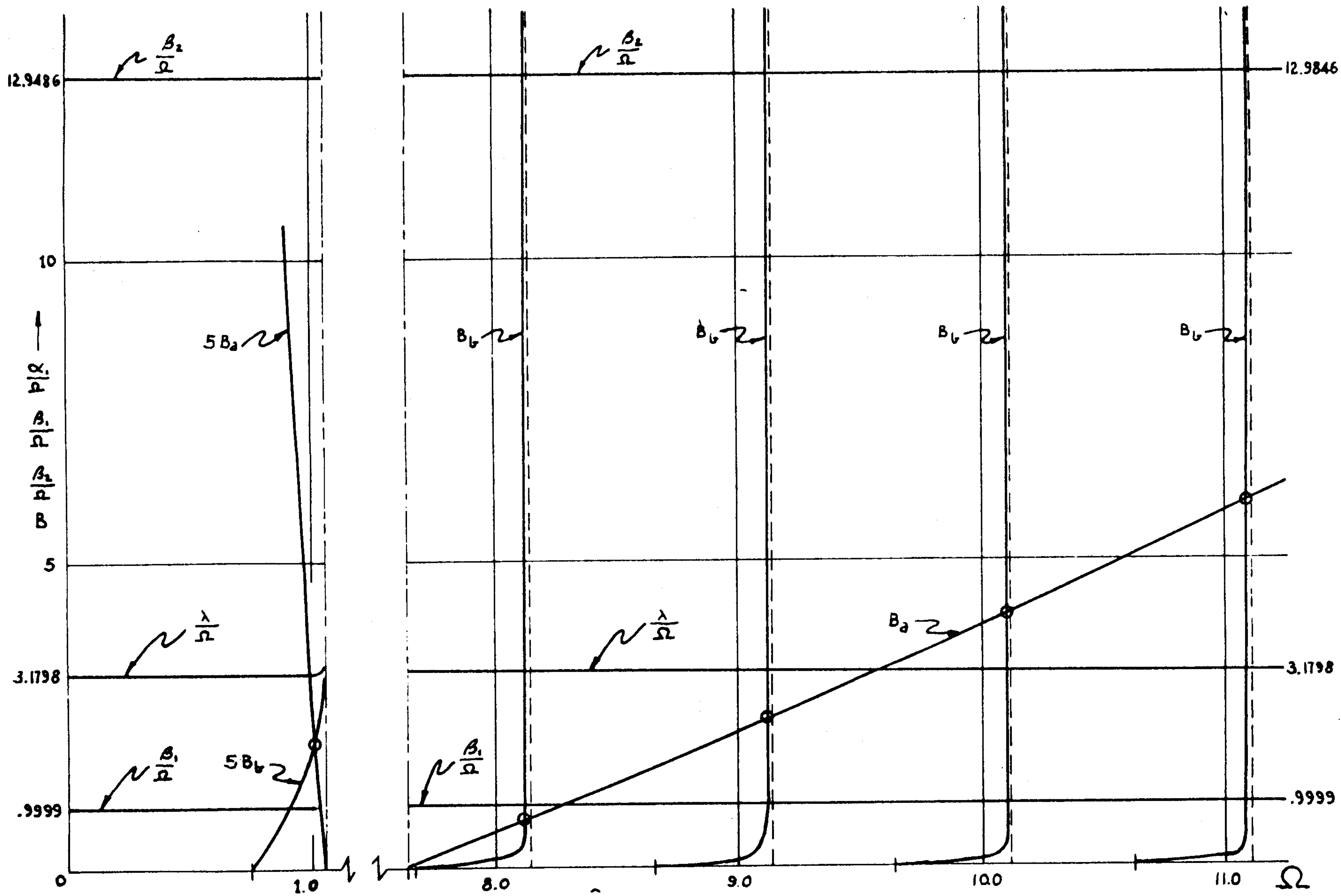

FIG.3 PARTIALLY FILLED TANK — AXISYMMETRIC CASE
SOLUTION OF CHARACTERISTIC EQUATION

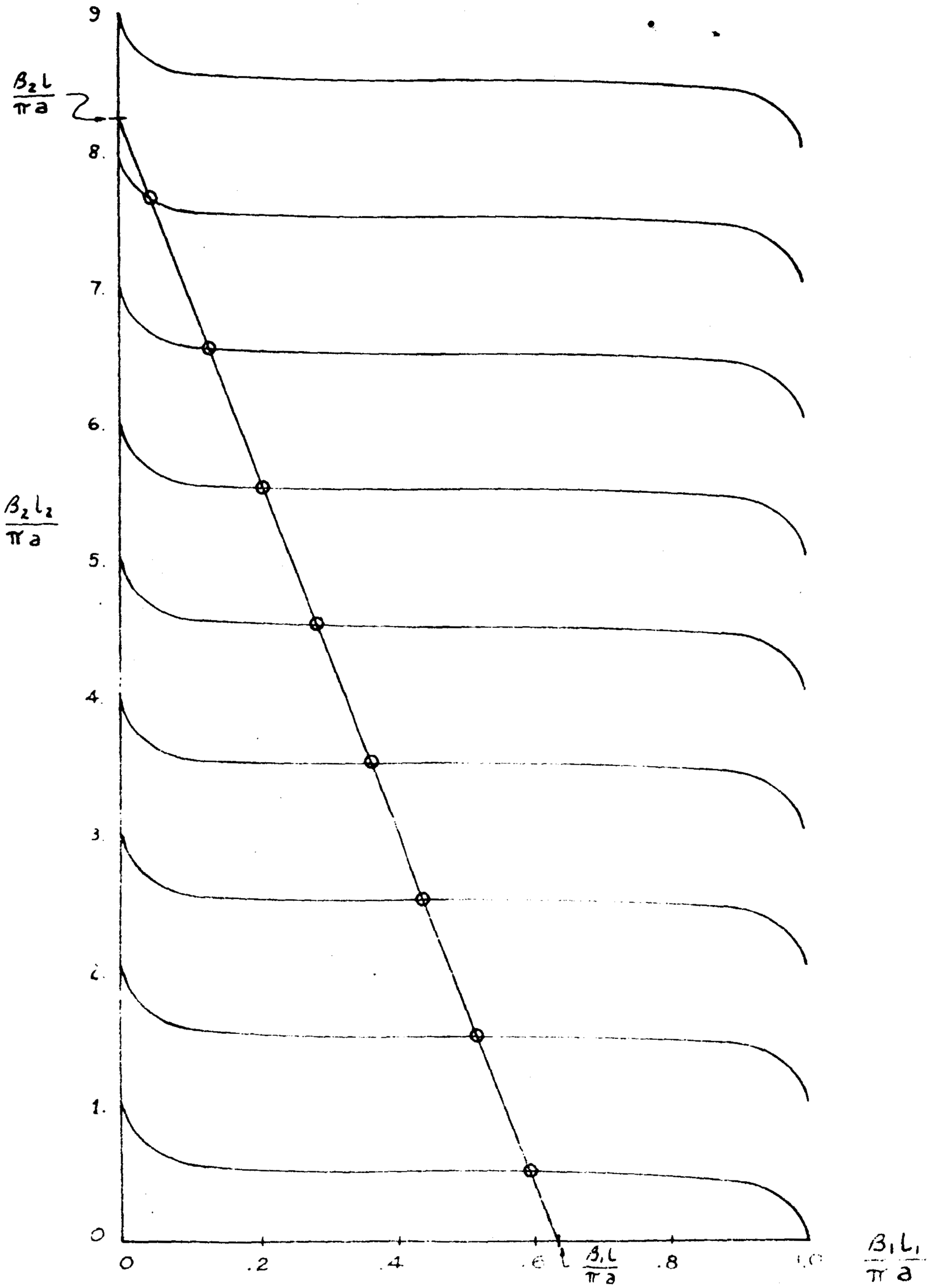

FIG. 4 PARTIALLY FILLED TANK—AXISYMMETRIC CASE
SOLUTION OF CONTINUITY EQUATION